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A FINITE FLAG-TRANSITIVE GEOMETRY OF EXTENDED G_2 -TYPE

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A finite Flag-Transitive Geometry of Extended G_2 -Type *)

by

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ABSTRACT

The purpose of this note is to give an explicit construction of a finite flag-transitive GAB with an extended G_2 -diagram having the group $G_2(3)$ as automorphism group. In our example it will be apparent that the intersection property is satisfied.

KEY WORDS & PHRASES: *Buekenhout-Tits geometries, geometries of extended G_2 type, Chevalley group $G_2(3)$*

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0. INTRODUCTION

Geometries that are almost buildings (GABs) were introduced by TITS in [11]. They are BUEKENHOUT-TITS geometries [1] in which all rank two residual geometries are generalized polygons, except they need not satisfy the intersection property. Tits has shown that they exist in great number, including finite ones. In [7] KANTOR remarks that the situation for finite GABs with large automorphism groups, other than those arising from buildings appear to be rare. In [7] KANTOR briefly describes four finite GABs having flag-transitive automorphism groups. The only other known finite flag-transitive GAB was constructed by RONAN and SMITH [9] from the Suzuki sporadic group. The purpose of this note is to give an explicit construction of a finite flag-transitive GAB with an extended G_2 -diagram having the group $G_2(3)$ as automorphism group. In our example it will be apparent that the intersection property is satisfied.

1. GEOMETRIES OF EXTENDED G_2 -TYPE

We will be concerned here with incidence structures $I = (P, L, \Pi; I)$ with three types of objects:

P , whose elements are called *points*;

L , whose elements are called *lines*; and

Π , whose elements are called *planes*, together with a symmetric relation

I on $E = P \cup L \cup \Pi$. We set $O = \{P, L, \Pi\}$.

Suppose $\{X, Y, Z\} = O$, $x \in X$. Set

$$Y_x = \{y \in Y \mid x I y\}, \quad Z_x = \{z \in Z \mid x I z\},$$

and

$$I_x = I \Big|_{Y_x \cup Z_x}.$$

$(Y_x, Z_x; I_x)$ is called the *residue* of I at x , I_x .

We say I is of *extended G_2 -type* or *belongs to the diagram $\bullet \text{---} \circ \equiv \circ$* if the following are satisfied

- (i) If $X \in \mathcal{O}$, $x \neq y \in X$, then x and y are not incident;
- (ii) If $x \in P$, I_x is a generalized hexagon;
- (iii) If $\ell \in L$, I_ℓ is a complete bipartite graph; and
- (iv) If $\pi \in \Pi$, I_π is a projective plane.

In our construction all residues will have order two.

2. THE GEOMETRY OF NON-ISOTROPIC POINTS IN THREE DIMENSIONAL UNITARY SPACE

Let $K = \mathbb{F}_q^2$, $\langle \tau \rangle = \text{Gal}(K/k)$, $k = \mathbb{F}_q$. Denote images under τ by $—$. Let V be a three-dimensional vector space over K and $h: V \times V \rightarrow K$ a non-degenerate hermitian form, that is, h should satisfy

- (i) For $v \in V$, $w \mapsto h_v(w) = h(w, v)$ is linear
- (ii) For $v, w \in V$, $h(w, v) = \overline{h(v, w)}$
- (iii) $h_v \equiv 0$ if and only if $v = 0$.

(V, h) is a *unitary space over K* .

Let $N = \{ \langle v \rangle \mid v \in V, h(v, v) \neq 0 \}$. N is the set of *non-isotropic points* of (V, h) . Define a graph on N as follows: for $p \neq q \in N$, $p \sim q$ if and only if $h(p, q) = 0$. Let Λ be the collection of maximal cliques in (N, \sim) . Clearly these are triples. We collect some facts about the partial linear space (N, Λ) .

$$(2.1) \quad |N| = q^2(q^2 - q + 1)$$

This is easy: $|PG(V)| = q^4 + q^2 + 1$. The number of isotropic or absolute points ($\langle x \rangle$ is absolute if $h(x, x) = 0$), is $q^3 + 1$.

For $x \in N$, set $\Gamma(x) = \{y \in N : x \sim y\}$. We let $d(,) = d_\Gamma(,)$ be the usual metric associated with (N, Γ) and $\Gamma_t(x)$ be the points at distance t from x .

$$(2.2) \quad |\Gamma(x)| = q^2 - q.$$

This is again easy: $x^\perp = \{v \in V : h(x, v) = 0\}$ is a non-degenerate two space, so $|PG(x^\perp)| = q^2 + 1$. The number of absolute points in x^\perp is $q + 1$.

Set $\Lambda_x = \{\lambda \in \Lambda : x \in \lambda\}$. Then clearly from (2.2) we have

$$(2.3) \quad |\Lambda_2| = (q^2 - q)/2$$

Next suppose $x \not\sim y$. There are clearly two possibilities:

- (i) $\langle x, y \rangle$ is non-degenerate; and
- (ii) $\text{Rad } \langle x, y \rangle = \langle x, y \rangle^\perp \cap \langle x, y \rangle$ is a (isotropic) point.

In case (i) we see that $x^\perp \cap y^\perp$ is a single point in N , while in (ii) $\langle x, y \rangle^\perp = \text{Rad } \langle x, y \rangle$. Thus in case (i), $d_\Gamma(x, y) = 2$ and in case (ii) $d_\Gamma(x, y) \geq 3$. A simple count yields

$$(2.4) \quad |\Gamma_2(x)| = (q^2 - q)(q^2 - q - 2).$$

Now it is not difficult to see that $O(V, h) = \{T \in GL(V) : h(Tv, Tw) = h(v, w)\}$ acts transitively on pairs $\{x, y\} \subseteq N$ with $d(x, y) = 2$, and on pairs with $\text{Rad } \langle x, y \rangle \neq 0$. It therefore follows that

$$(2.5) \quad (N, \Gamma) \text{ is distance transitive with diameter } 3.$$

We now turn our attention to pairs $\{x, y\}$ with $\text{Rad } \langle x, y \rangle \neq 0$. Let $x = \langle v \rangle$, $y = \langle w \rangle$ where $h(v, v) = h(w, w) = 1$. Suppose $\lambda = \{x = x_1, x_2, x_3\}$ is a line on x . Let $v = v_1$, $x_i = \langle v_i \rangle$ where $h(v_i, v_i) = 1$, $i = 2, 3$. Now set $R = \text{Rad } \langle x, y \rangle$. Then $w = a v + r$ where $r \in R$. Since $h(w, w) = h(v, v) = 1$, without loss of generality we may assume $a = 1$. Since $h(v, r) = 0$ and $v^\perp = x^\perp = \langle x_2, x_3 \rangle$, there are $b, c \in K$ so $r = b v_2 + c v_3$. Since $h(r, r) = 0$ we must have

$$(2.6) \quad b\bar{b} + c\bar{c} = 0.$$

We will determine conditions for $d(y, x_i) = 2$ for $i = 2, 3$. Now $d(y, x_i) = 2$ if and only if $y^\perp \cap x_i^\perp \in N$. $x_i^\perp = \langle x_1, x_j \rangle$ where $\{i, j\} = \{2, 3\}$. Note if $\alpha v_1 + \beta v_j \in y^\perp$, then $\alpha \beta \neq 0$, so we may take $\beta = 1$. If $h(\alpha v_1 + v_j, v_1 + b v_2 + c v_3) = 0$ then

$$\alpha = \begin{cases} -\bar{b} & j = 2 \\ -\bar{c} & j = 3 \end{cases} \quad \text{Set } \alpha_j = \begin{cases} -\bar{b} & j = 2 \\ -\bar{c} & j = 3 \end{cases}$$

Now $h(\alpha_j v_1 + v_j, \alpha_j v_1 + v_j) = \alpha_j \bar{\alpha}_j + 1$. Now if we assume $\text{char}(k) \neq 2$, then by (2.6) we cannot have $\alpha_2 \bar{\alpha}_2 + 1 = 0 = \alpha_3 \bar{\alpha}_3 + 1$. We have therefore shown

$$(2.7) \quad \text{If } d(x, y) = 2, \lambda \in \Lambda_x, \text{ then } \lambda \cap \Gamma_2(y) \neq \emptyset.$$

Now if $q > 3$ we can easily see that there are pairs x, y with $d(x, y) = 3$ and lines λ on x such that $\lambda - \{x\} \subseteq \Gamma_2(y)$. However, from (2.6) we see that

$$(2.8) \quad \text{If } q = 3, y \in \Gamma_3(x), \lambda \in \Lambda_x, \text{ then } |\lambda \cap \Gamma_2(y)| = 1.$$

We have thus demonstrated all we need for

(2.9) THEOREM. *The geometry (N, Λ) is a generalized hexagon if, and only if, $q = 3$.*

(2.10) REMARK. When $q = 3$ the generalized hexagon (N, Λ) is the dual of the usual $(2, 2)$ -generalized hexagon associated with $G_2(2)$. [The usual $G_2(2^n)$ hexagon is the one embeddable in $\text{PG}(5, 2^n)$].

This follows from the fact that if $Q = O_2(G_x), x \in N$, then Q' , the commutator subgroup of Q , has order two.

3. THE OCTAVES AND THE G_2 -GENERALIZED HEXAGON

Let k be a commutative field, $\mathbb{O}(k) = \mathbb{O}$ the split octaves over k . \mathbb{O} is a *composition algebra*, that is \mathbb{O} is an algebra with identity and admits a non-degenerate quadratic form Q such that $Q(x \cdot y) = Q(x)Q(y)$. We can find an orthonormal base $1 = e_0, e_1, \dots, e_7$ for \mathbb{O} , with e_0 the identity element such that multiplication in \mathbb{O} is determined by

$$(3.1) \quad e_i^2 = -1, \quad 1 \leq i \leq 7$$

and

$$(3.2) \quad e_i e_j = -e_j e_i = e_k \quad \text{whenever } (ijk) \text{ is one of the three cycles } (1+2, 2+4, 4+1) \text{ where } i, j, k, r \text{ run through the integers modulo seven and take their values in } \{1, 2, \dots, 7\}.$$

Let $W = e_0^\perp = \langle e_1, \dots, e_7 \rangle$, so that $Q|_W$ is a non-degenerate quadratic form (with maximal Witt index). Let $G = \text{Aut}(\mathbb{O})$. Clearly G leaves e_0 and W invariant.

It is well-known (cf.[5]) that G is the Chevalley group $G_2(k)$. Moreover, it is well known that there is, up to isomorphism, only one such algebra \mathcal{O} (cf.(3.1) in [5]). Consequently we have

(3.3) If f_1, \dots, f_7 satisfy (3.1), (3.2) then there exist $\sigma \in \text{Aut}(\mathcal{O})$ with $\sigma f_i = e_i, 1 \leq i \leq 7$.

Of interest to us will be the set $\Phi = \{e_i, -e_i, 1 \leq i \leq 7\}$, and its full stabilizer, G_Φ , in G . This group is also well-known (see [3] and [4]):

(3.4) G_Φ is a non-split extension of an elementary abelian group E of order 8 by $\text{PSL}_3(2)$.

For the sequel we assume that -1 is not a square in k . Then $K = \langle e_0, e_1 \rangle = k(e_1) \cong k[t]/(t^2+1)$ is a quadratic extension of k .

Set $V = \langle e_2, \dots, e_7 \rangle = W \cap e_1^\perp$. Note that V becomes a three dimensional vector space over K by restriction of the multiplication of \mathcal{O} to $K \times V$.

For $u = ae_0 + be_1 \in K$, set $\bar{u} = ae_0 - be_1$. Then $\bar{}$ generates the Galois group of K over k . Next define $h: V \times V \rightarrow K$ to be $p \circ \mu|_{V \times V}$, where μ is the multiplication of \mathcal{O} and p is the projection of \mathcal{O} onto K . Then

(3.5) h is a non-degenerate hermitian form on V with associated automorphism $\bar{}$. Moreover, $\sigma \in G_{\langle e_1 \rangle}$ if and only if $\sigma|_V$ is a unitary transformation, i.e. preserves h .

4. THE CONSTRUCTION OF THE EXTENDED G_2 -GEOMETRY OF ORDER 2

We retain the notation of the previous sections. Further we set

$$P = \langle e_1 \rangle^G = \{ \langle w \rangle : w \in W, Q(w) = 1 \}$$

$$L = \ell^G, \text{ where } \ell = \{ \langle e_1 \rangle, \langle e_2 \rangle, \langle e_4 \rangle \}; \text{ and}$$

$$\Pi = \pi^G \text{ where } \pi = \{ \langle e_i \rangle : 1 \leq i \leq 7 \}.$$

Let I be symmeterized inclusion of subsets of P restricted to $\bar{E} = P \cup L \cup \Pi$. We will prove

(4.1) THEOREM: $I = (P, L, \Pi; I)$ is a geometry of extended G_2 -type if and only if $k = \mathbb{F}_3$.

We proceed to prove this in a series of steps. Note that $G = \text{Aut}(\emptyset)$ is flag-transitive by (3.3). Because of this it suffices to check the residues $I_{\langle e_1 \rangle}$, I_ℓ , I_π .

(4.2). I_π is a projective plane of order 2.

pf: $\{\langle e_i \rangle, \langle e_j \rangle, \langle e_k \rangle\} \in L_\pi$ for all triples (ijk) as in (3.2). If these are all the lines in L_π , then the result is clear. Now $C = G_\Phi \leq G_\pi$ and is two transitive on π . Moreover, for any $i \neq j$ $C_{\langle e_i \rangle, \langle e_j \rangle}$ fixes $\langle e_k \rangle$ where (ijk) is as in (3.2) and is transitive on $\pi - \{\langle e_i \rangle, \langle e_j \rangle, \langle e_k \rangle\}$. It therefore suffices to prove $\{\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle\} \notin L$. But this is obvious since $G \text{ tra } L$ and elements of G preserve multiplication.

(4.3) $I_{\langle e_1 \rangle}$ is a generalized hexagon if and only if $k = \mathbb{F}_3$.

pf: Let $\ell \in L_{\langle e_1 \rangle}$. Define $\theta(\ell) = \ell \cap e_1^\perp$. Now $\ell \in L_{\langle e_1 \rangle}$ if and only if the subalgebra generated by e_0 and ℓ is $ke_0 + k\ell$, and this occurs if and only if $\theta(\ell)$ is a one dimensional non-isotropic subspace of the unitary K -space $V = W \cap e_1^\perp$. Thus $\theta(L_{\langle e_1 \rangle}) = N(V, h)$, the non-isotropic points. Because of the flag transitivity we see that $\ell_1, \ell_2 \in L_{\langle e_1 \rangle}$ lie in a common plane if and only if the points $\theta(\ell_1), \theta(\ell_2)$ of V are orthogonal. Thus $I_{\langle e_1 \rangle}$ is isomorphic to the geometry (N, Λ) of section two. By (2.10) $I_{\langle e_1 \rangle}$ is a generalized hexagon if and only if $k = \mathbb{F}_3$, and in this case $I_{\langle e_1 \rangle}$ is the dual of the usual $(2,2)$ -hexagon.

(4.4) If $k = \mathbb{F}_3$ then I_ℓ is the complete bipartite graph $K_{3,3}$.

pf: Clearly I_ℓ is a complete bipartite graph and as $|\ell| = 3$ it suffices to prove that there are three planes containing ℓ . However, if $\pi' \in \Pi_\ell$, then $\pi' \in \Pi_{\langle e_1 \rangle}$ and π' is a line of $I_{\langle e_1 \rangle}$ containing ℓ . Since $I_{\langle e_1 \rangle}$ has order $(2,2)$ it follows that there are precisely three planes containing ℓ .

This completes the proof of the theorem.

5. CONCLUDING REMARKS

Following KANTOR [4] we define an apartment to be a subset Δ of P such that there are automorphisms r, s, t of I leaving Δ invariant such that $r^\Delta, s^\Delta, t^\Delta$ satisfy the relations $\cdot \text{---} \cdot \equiv \cdot$ while $\langle r, s, t \rangle^\Delta$ acts flag-transitively. We have

(5.1) PROPOSITION. *Apartments do not exist.*

PROOF. Suppose that Δ, r, s, t exist. Then there is a point $P_0 \in \Delta$ fixed by s and t and points P_1, \dots, P_6 in Δ all collinear with P_0 such that P_i is collinear with P_{i+1} (modulo 6). Moreover triangles in Δ are not lines of I . However, $\langle s^\Delta, t^\Delta \rangle$ contains an elementary group of order 4 which cannot act regularly on $\{P_1, \dots, P_6\}$, therefore there is an involution τ in $\langle s, t \rangle$ fixing at least two of $\{P_1, \dots, P_6\}$. But then since τ fixes P_0 , $|\text{Fix}_\Delta(\tau)| \geq 3$. However, for any involution in G , $\text{Fix}_\Delta(\tau)$ is a line. This contradicts the fact that Δ contains no full lines.

(5.2) REMARK. In [6] Goldschmidt studied groups G generated by a pair of subgroups P_1, P_2 where P_1, P_2 contain a common 2-Sylow S of G and $|P_i:S| = 3$, $i = 1, 2$, and determined all such groups. In analogy with the groups of Lie type we might call such subgroups P_i parabolics. This theory has been extended to the case $|P_1:S| = q + 1$, $P_i/O_2(P_i) \cong L_2(q)$. CHERMAK [2], TIMMESFELD [10] and others have considered the problem of determining groups $G = \langle P_1, \dots, P_n \rangle$ with $n \geq 3$ such that all P_i contain a common 2-Sylow S , $|P_i:S| = q + 1$ and $P_i/O_2(P_i) \cong \text{PSL}_2(q)$, q a power of two, especially the case where for any $i \neq j$, $\langle P_i, P_j \rangle / O_2(\langle P_i, P_j \rangle) \cong L_2(q) \times L_2(q)$ or $L_3(q)$ or $\text{SL}_3(q)$. In general the resulting group is a Chevalley group of type A_n, D_n, E_n over a field of characteristic two. Our construction provides an example of a group G generated by three "parabolics" P_1, P_2, P_3 containing a common two Sylow S with $|P_i:S| = 3$ and such that for any $i \neq j$, $\langle P_i, P_j \rangle / O_2(\langle P_i, P_j \rangle)$ is one of $\text{PSL}_3(2)$, $\text{PSL}_2(2) \times \text{PSL}_2(2)$, $G_2(2)$. This suggests that a general classification will prove extremely difficult.

(5.3) FINAL REMARK. In some sense the existence of such a geometry for $G_2(3)$ should not be surprising: A result of MASON'S [8] classifies the groups G of characteristic 2,3-type under very mild hypotheses. The groups are $\text{PSP}_4(3)$, $U_4(3)$ and $G_2(3)$. Of course $\text{PSP}_4(3)$ is not particularly exceptional because of the isomorphism $\text{PSP}_4(3) \cong \Omega_6^-(2)$. In [7] KANTOR constructs a GAB for $U_4(3)$, and our construction is for $G_2(3)$. Thus these groups act flag-transitively on \mathbb{F}_3 -buildings and as flag-transitive groups on \mathbb{F}_2 -'near' buildings.

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